

**LIMITS IN THE COMPLEX PLANE**

**1 Definition**

Limits in the complex functions are very similar to limits in the real functions, the same formal definitions ( $\epsilon - \delta$ ) still hold and all the same rules such as addition and multiplication of limits can still be used. The only difference is whilst in the real axis there are only two directions to approach an limit, in the complex plane there are infinite ways to approach a limit.

To show that a limit exists in the complex plane, we need to show that the  $\lim_{z \rightarrow z_0} f(z)$  is the same from every direction. To show that a limit does not exist, we only need to find two directions of the limit that are not the same.

**2 Limits along the horizontal or vertical lines**

When showing that a limit does not exist, the easiest way to to calculate the limit along the vertical and horizontal lines. We do this by fixing either the real part or the imaginary part and then evaluating the limit along the other part.

**2.1 Example**

For  $f(z) = z \operatorname{Im} z$ , use the limit definition of the derivative to show where  $f'(z)$  exists.

Solution

The limit definition of derivative is

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Put  $z = x + iy$  and  $z_0 = x_0 + iy_0$ . On  $x = x_0$  (along the vertical line)

$$L = \lim_{y \rightarrow y_0} \frac{z \operatorname{Im} z - z_0 \operatorname{Im} z_0}{z - z_0} = \lim_{y \rightarrow y_0} \frac{(x_0 + iy)(y) - (x_0 + iy_0)(y_0)}{x_0 + iy - (x_0 + iy_0)}.$$

Rearranging this gives

$$L = \lim_{y \rightarrow y_0} \frac{x_0(y - y_0) + i(y^2 - y_0^2)}{i(y - y_0)}.$$

This can be split into real and imaginary parts to give

$$L = \lim_{y \rightarrow y_0} \left[ \frac{x_0(y - y_0)}{i(y - y_0)} + \frac{i(y^2 - y_0^2)}{i(y - y_0)} \right] = \lim_{y \rightarrow y_0} \frac{x_0}{i} + \lim_{y \rightarrow y_0} \frac{y^2 - y_0^2}{y - y_0},$$

$$L = -ix_0 + \lim_{y \rightarrow y_0} y + y_0 = -ix_0 + 2y_0.$$

Now we do the exact same method on  $y = y_0$  (along the horizontal line).

$$L = \lim_{x \rightarrow x_0} \frac{z \operatorname{Im} z - z_0 \operatorname{Im} z_0}{z - z_0} = \lim_{x \rightarrow x_0} \frac{(x + iy_0)(y_0) - (x_0 + iy_0)(y_0)}{x + iy_0 - (x_0 + iy_0)}.$$

Rearranging this gives

$$f'(z_0) = \lim_{x \rightarrow x_0} \frac{(x - x_0)y_0}{(x - x_0)} = y_0.$$

From this we can tell from the limit definition that the derivative does not exist when  $y_0 \neq -ix_0 + 2y_0$ . We cannot say for sure that the derivative does exist when  $y_0 = x_0 = 0$  because we have only taken two directions, to prove it does exist we must use a method that considers all possible directions.

### 3 Limits in polar form

When finding the limit as  $z \rightarrow z_0$ , by letting  $z = z_0 + r e^{i\theta}$  we can consider every possible direction into  $z_0$  by finding the limit as  $r \rightarrow 0$  and keeping  $\theta$  as arbitrary. This is an extremely helpful technique and is especially useful when  $z_0 = 0$ . Here we will use polar form to prove that the derivative in the previous example exists at  $z = 0$ .

#### 3.1 Example

For  $f(z) = z \operatorname{Im} z$ , use the limit definition of the derivative to determine of  $f'(0)$ .

##### Solution

As in the previous example, we use the limit definition of the derivative on this example.

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{z \operatorname{Im} z - z_0 \operatorname{Im} z_0}{z - z_0}$$

Now we convert into polar form, and substitute  $z_0 = 0$  to get

$$L = \lim_{r \rightarrow 0} \frac{r e^{i\theta} \operatorname{Im}(r e^{i\theta})}{r e^{i\theta}}$$

We can cancel out most terms in this expression, and we note that the imaginary part of  $r e^{i\theta}$  is  $r \sin \theta$

$$L = \lim_{r \rightarrow 0} r \sin \theta$$

The next step we use the squeeze theorem to show that

$$|r \sin \theta| = |z| |\sin \theta| \leq |r| = r$$

Since  $|r| \rightarrow 0$  as  $r \rightarrow 0$ , we can say by the squeeze theorem that  $L = 0$ . Since this is considering every possible path into the origin we can determine that

$$f'(0) = 0$$

### 4 Key points

- Limits in polar form are the best way to prove a limit exists, but can be difficult to calculate unless the limit is taken at the origin. The straight line method is usually the easiest to show where the limit does not exist.
- For calculating limits as  $z \rightarrow z_0$  along the horizontal line, fix  $y$  to be  $y_0$  then calculate the limit as  $x \rightarrow x_0$ . For calculating limits along the vertical line, do the opposite.
- When calculating limits in polar form, remember to change the limit so you are calculating as  $r \rightarrow 0$ , not as  $z \rightarrow z_0$ .

*For more information on limits in the complex plane refer to the lecture notes.*